All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means for a function $f : \mathbb{C} \to \mathbb{C}$ to be *holomorphic* at the point z_0.

   (b) Show that $f(z) = \bar{z}$ is not holomorphic at z_0 for any $z_0 \in \mathbb{C}$.

   (c) Write the Cauchy-Riemann equations for u and v, where $f(z) = u(x, y)+iv(x, y)$ i.e. u and v are the real and imaginary part of f. Show that, if f is holomorphic, then u is harmonic.

   You may assume that the second partial derivatives of u and v exist and are continuous.

   (d) For the harmonic function $v : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by the formula

   $$v(x, y) = \frac{-y}{x^2 + y^2}$$

   find all holomorphic functions $f(z)$ such that $\Im f(z) = v(x, y)$. Write f as a function of z.

2. (a) Assume that f is holomorphic on the domain D and that $|f(z)|$ is constant on D. Show that f is a constant function.

   You may use the fact that if $f'(z) = 0$ on D, then f is constant.

   (b) Establish the following integration formula with the aid of residues:

   $$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{6}.$$

   Complete explanations are required.

3. (a) State Goursat's theorem.

   (b) Let f be a holomorphic function on a domain D containing a rectangle R and its interior. Using Goursat's theorem, show that

   $$\int_R f(z)\, dz = 0.$$

   (c) Show that the map $w = \frac{z-i}{z+i}$ maps the upper half-plane $\{z; \Im(z) > 0\}$ conformally onto the unit disc $\{z; |z| < 1\}$.

4. (a) State Cauchy's integral formulas for f and its derivatives.

(b) What is the value of the integral

$$\int_C \frac{1}{z^2+1}\,dz,$$

where C is (i) the circle $|z| = 2$ traversed anticlockwise, (ii) the circle $|z-i| = 1$ traversed anticlockwise?

(c) Assume that f is entire and satisfies for some constant M the inequality

$$|f(z)| \le M(1+|z|)^{5/2}, \quad \forall z \in \mathbb{C}.$$

Show that f is a polynomial of degree at most 2.

5. (a) How many roots does the polynomial $f(z) = 2z^5 - 6z^2 + z + 1$ have inside the annulus

$$1 < |z| < 2?$$

Explain your answer.

(b) Establish the following integration formula with the aid of residues:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = -\frac{\pi \sin 2}{e}.$$

Complete explanations are required.

6. (a) Let $f(z)$ be holomorphic on an open set that contains the closed unit disc $\{z; |z| \le 1\}$, with $f(0) = 1/2$. By working with

$$\frac{1}{2\pi i}\int_{|z|=1} \left[2 \pm \left(z + \frac{1}{z}\right)\right] f(z)\frac{dz}{z}$$

prove that

$$\frac{2}{\pi}\int_0^{2\pi} f(e^{it})\cos^2\frac{t}{2}\,dt = 1 + f'(0), \quad \frac{2}{\pi}\int_0^{2\pi} f(e^{it})\sin^2\frac{t}{2}\,dt = 1 - f'(0).$$

(b) Establish the following integration formula with the aid of residues:

$$\int_0^{2\pi} \frac{5}{5+3\cos t}\,dt = \frac{5\pi}{2}.$$

Complete explanations are required.